Uniform Scalar Quantizers – Distortion and the Additive Noise Model

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Abstract — **High-resolution infinite level uniform scalar quantizers are considered for the two cases when cell midpoints and cell centroids are used as reconstruction levels. First, it is shown that asymptotically, as the cell size goes to zero, midpoints induce the same MSE as centroids. Hence, they are asymptotically MSE optimal. Secondly, sufficient conditions are found under which the commonly used additive noise model, when midpoints are used, is asymptotically correct. It is also shown that for input probability density functions (pdf's) that have a single jump discontinuity not at the origin, such a model is not well defined and consequently incorrect.**

I. Notation

We consider an infinite level scalar quantizer with step size Δ and thresholds $\{..,-\Delta,0,\Delta,2\Delta,..\}$[2] Let X denote the input random variable. If $[a,a+\Delta)$ is a quantization cell, then its midpoint is given by $a+\frac{\Delta}{2}$, and its centroid is given by $E[X|a \le X < a+\Delta]$. For a particular step size Δ, let $M_\Delta, D_{m,\Delta}$ and $C_\Delta, D_{c,\Delta}$ denote the output and distortion of the quantizer, when midpoints and centroids are used, respectively. For brevity, we will omit the Δ subscript, except where it is needed for emphasis.

II. Distortion

It is well-known from high-resolution theory that the distortion when using midpoints is approximately $\Delta^2/12$. Indeed, Linder and Zeger [1] showed rigorously that for any pdf,

$$D_{m,\Delta} = \frac{\Delta^2}{12} + o(\Delta^2) \tag{1}$$

It is also well-known that using centroids minimizes MSE. Our first result is that for inputs with piecewise continuous pdf's, centroids asymptotically induce the same MSE as midpoints. Specifically,

$$D_{c,\Delta} = D_{m,\Delta} + o(\Delta^2) \tag{2}$$

It follows that midpoints are asymptotically optimal in terms of minimizing MSE.

III. Additive Noise Model

Under high-resolution conditions, Bennett [2] showed that when the input pdf is "smooth", in some intuitive but unspecified sense, the quantization error may be modeled as additive noise. Namely, the quantization error is approximately orthogonal to the input, i.e.

$$EX(X-M) \approx 0 \quad \text{and} \quad EM^2 \approx EX^2 + D_m \tag{3}$$

Although the additive model is very frequently used [3], to the best of our knowledge the precise conditions under which it is valid have yet to be stated. While it is easy to see that the difference between the left and right hand sides of (3) goes to zero as $\Delta \to 0$, what in fact needs to be shown is that

$$EM^2 = EX^2 + D_m + o(\Delta^2) \tag{4}$$

(Recall from (1) that D_m is of the order of Δ^2.) More specifically we have,

$$EM^2 = E(X-(X-M))^2 = EX^2 + D_m - 2EX(X-M) \tag{5}$$

and it is now clear from (4) and (5) that the input and quantization error have to be asymptotically orthogonal in the following sense,

$$\lim_{\Delta\to 0} \frac{EX(X-M)}{\Delta^2} = 0 \tag{6}$$

Alternatively, since $EC^2 = EX^2 - D_c$ (due to the orthogonality principle) and using (2), the power of the output may be written as follows,

$$EM^2 = EX^2 - D_m + (EM^2 - EC^2) + o(\Delta^2) \tag{7}$$

Therefore, an equivalent condition for the additive noise model to be correct is that $EM^2 - EC^2 \approx 2D_m$ or more precisely,

$$\lim_{\Delta\to 0} \frac{EM^2 - EC^2}{\Delta^2/6} = 1 \tag{8}$$

Our second result, which is composed of two statements, gives conditions under which the above holds and under which the above is not well defined.

Positive statement: If the input pdf f, is continuously differentiable with bounded derivative (and possibly one discontinuity at the origin) such that $lim_{x\to-\infty}|x|^{2+\delta}f'(x) = 0$ and $lim_{x\to\infty}x^{2+\delta}f'(x) = 0$ for some $\delta > 0$, then (8) holds. Consequently, the additive noise model is correct.

Negative statement: If the input pdf has a single jump discontinuity not at the origin, then the limit in (8) is not well defined. Consequently, the additive noise model is incorrect.

One step in proving the positive statement is showing that the stated conditions are sufficient to permit the swapping of limit and expectation in (8). A second step is finding an accurate asymptotic expression for cell centroids.

References

[1] T. Linder, K. Zeger, "Asymptotic Entropy-Constrained Performance of Tessellating and Universal Randomized Lattice Quantization," *IEEE Tr. Info. Thy.*, vol.40, pp. 575-579, Mar. 1994.

[2] W.R Bennett, "Spectra of quantized signals," *Bell Syst. Tech. J.*, vol. 27, pp. 446-472, Jul. 1948.

[3] A. V. Oppenheim, R. W. Schafer and J.R. Buck, *Discrete-Time Signal Processing*, 2nd edition, Prentice-Hall 1999.

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[2]Our results extend to the case where the thresholds have the form $(i+s)\Delta$ for some $s \in (0,1)$.